



<b>Title</b>	<b>Distributed event-triggered control for output synchronization of dynamical networks with non-identical nodes</b>
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# Distributed Event-Triggered Control for Output Synchronization of Dynamical Networks with Non-identical Nodes

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**Abstract**—This paper studies the output synchronization problem of a dynamical network with event-based communication, where each node communicates to its neighbours only when an event-triggering condition is fulfilled. The network has non-identical nodes driven by identical exosystems. In order to achieve asymptotic output synchronization as well as to prevent the occurrence of Zeno behaviour, estimators are introduced into each node to estimate the states of the exosystems of its neighbours and its own. Then, a distributed event-triggering rule is designed, which only depends on the information that the node obtains from its neighbours and the states of introduced estimators. Finally, a numerical example is given to show the effectiveness of the proposed control.

## I. INTRODUCTION

Synchronization of dynamical networks or interconnected systems and its related problem – consensus of multi-agent systems have attracted a great deal of attention due to their extensive applications in physics, biology and engineering [1], [2], [3]. Motivated by the fact that connected nodes in some real-world networks share information over a digital platform, synchronization and consensus problems have also been investigated under the circumstance that nodes or agents communicate with each other only at some discrete time instances that are based on the occurrence of a well-defined event, i.e., to achieve synchronization or consensus by designing event-triggered controllers [4].

In [5], a distributed event-triggered control mechanism was developed to investigate asymptotic consensus of a multi-agent system. This control method was further extended to the study of  $L_2$  gain stability of the system with additive disturbances in [6]. To guarantee asymptotic consensus, as well as prevent the occurrence of Zeno behaviour, a threshold exponentially decreasing in time was introduced into a decentralized event-triggering rule in [7]. Most recently, for a network with generalized linear node dynamics, a distributed event-triggered control method was introduced in [8], under which asymptotic synchronization of the network can be achieved. But there is no evidence that the designed event-triggering rule can prevent Zeno behaviours. Moreover, in

[9], estimators were introduced into each node, and were used to design a decentralized event-triggering rule with a fixed threshold, but only bounded synchronization was obtained. To overcome issues encountered in [8] and [9], the authors in [10] used estimators similar to [9] to design a new distributed event-triggered rule such that the network achieves synchronization asymptotically without Zeno behaviours occurring.

All works mentioned above only focused on dynamical networks or multi-agent systems with identical nodes. In practice, most of real-world networks have non-identical nodes, and for these networks state synchronization is normally impossible. For this reason, output synchronization of networks with non-identical nodes has attracted a lot of attention in recent years [11], [12], [13], [14], [15]. In [15], the synchronization problem for non-identical nodes was studied in terms of bounded synchronization. In particular, a framework for output synchronization of networks with non-identical nodes driven by identical exosystems was proposed in [11], where the problem was solved in two steps: 1). design a distributed controller to achieve synchronization of the identical exosystems; 2). design a decentralized controller to force the output of each node to track the synchronized output of the corresponding exosystem. However, these works only studied networks with continuous interconnections with respect to time, and as far as we know, there are no works studying the output synchronization problem of dynamical networks with non-identical nodes by designing event-triggered controllers.

In view of the above results, we adopt the idea used in [11] and utilise the method proposed in [10] to study asymptotic output synchronization of a dynamical network with non-identical nodes by designing a distributed event-triggering rule. First, estimators are introduced into each node to generate continuous estimations of the exosystems of its neighbours as well as its own. Then, a distributed event-triggered controller is designed to achieve synchronization of the identical exosystems. At last, output synchronization of the network is achieved by designing a decentralized controller which makes the output of each node track the output of the corresponding exosystem. The designed event-triggered controller only uses information that each node obtains from its neighbours as well as the states of the estimators built in it. It is shown that under the designed controllers the network achieves output synchronization asymptotically, and no Zeno behaviour occurs.

The rest of the paper is organized as follows. Section II introduces the network model and some preliminaries.

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Section III studies output synchronization of the network by designing centralized and distributed event-triggered controllers. Section IV gives a numerical example to illustrate the obtained results. Conclusions are addressed in Section V.

*Notation:* We denote by  $\mathbb{R}$ ,  $\mathbb{R}^+$  and  $\mathbb{Z}^+$  the set of real numbers, non-negative numbers and non-negative integers, respectively; by  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times m}$  the set of  $n$ -dimensional real vectors and  $n \times m$  real matrices.  $I_n$ ,  $1_n$  and  $1_{n \times m}$  are the  $n$ -dimensional identity matrix, the  $n$ -dimensional vector with all entries being 1 and the  $n \times m$  matrix also with all entries being 1, respectively. We use  $\|\cdot\|$  to represent the Euclidean norm of a vector  $x \in \mathbb{R}^n$  or the induced norm of a matrix  $A \in \mathbb{R}^{n \times m}$ . The superscript “ $\top$ ” is the transpose of a vector or a matrix, and “ $^{-1}$ ” is the inverse of a nonsingular matrix. We denote the Kronecker product of two matrices by  $\otimes$ , and denote all the eigenvalues of a square matrix by  $\lambda(\cdot)$ .  $\text{Re}(\cdot)$  represents the real part of a complex number.

## II. NETWORK MODEL AND PRELIMINARIES

Consider a dynamical network with non-identical nodes driven by identical exosystems. The state equation of each node is given as follows

$$\dot{x}_i = A_i x_i + B_i u_{x_i} + E_i v_i, \quad (1a)$$

$$y_i = C_i x_i, \quad (1b)$$

$$\dot{v}_i = H v_i + u_{v_i} \quad i = 1, 2, \dots, N, \quad (1c)$$

where  $x_i \in \mathbb{R}^{n_i}$ ,  $u_{x_i} \in \mathbb{R}^{m_i}$  and  $y_i \in \mathbb{R}^{p_i}$  are the state, input and output of node  $i$ , respectively.  $v_i \in \mathbb{R}^{n_v}$  is the external input signal of node  $i$  governed by identical exosystem (1c), and  $u_{v_i} \in \mathbb{R}^{m_v}$  is the input of  $v_i$ .  $A_i \in \mathbb{R}^{n_i \times n_i}$ ,  $B_i \in \mathbb{R}^{n_i \times m_i}$ ,  $E_i \in \mathbb{R}^{n_i \times n_v}$ ,  $C_i \in \mathbb{R}^{p_i \times n_i}$  and  $H \in \mathbb{R}^{n_v \times n_v}$  are constant matrices.

We assume that the network is connected via a communication network, and each node can only access information from neighbours at certain discrete time instances. Therefore, the problem to be considered is as follows: with the given network topology and the information that each node can obtain, how to design the control inputs  $u_{x_i}$ ,  $u_{v_i}$  and to determine the time sequence  $\{t_{k_i}^i\}$ ,  $k_i \in \mathbb{Z}^+$  when node  $i$  should communicate to its neighbours (i.e., sample the state  $v_i(t)$  and send the sampled value  $v_i(t_{k_i}^i)$  to its neighbours) such that network (1) can achieve output synchronization asymptotically.

We adopt event-triggered control, i.e., design an event-triggering rule  $r_i(t)$  to determine such a time sequence  $\{t_{k_i}^i\}$ . Under this circumstance, Zeno behaviour may occur [7]. So it is also important to exclude the occurrence of such behaviour by designing proper control inputs  $u_{x_i}$ ,  $u_{v_i}$ , and most importantly a well-defined event-triggering rule  $r_i(t)$  for each node  $i$ . This is the main purpose of this paper.

*Remark 1:* As we focus on the design of a distributed event-triggered controller, we only consider a simplified network model (1) with linear node dynamics which can be considered as a linear version of the network model investigated in [11]. However, the issue of designing a distributed event-triggered controller for a general nonlinear network deserves attention in the future.

To solve this problem, we use estimators proposed in [10] and design inputs  $u_{x_i}$  and  $u_{v_i}$  as follows

$$\begin{aligned} u_{x_i} &= K_{x_i} x_i + K_{v_i} v_i \\ u_{v_i} &= c \sum_{j \in \mathcal{N}_i} a_{ij} (\hat{v}_j^i(t) - \hat{v}_i^i(t)), i = 1, 2, \dots, N, \end{aligned} \quad (2)$$

where  $K_{x_i}$  and  $K_{v_i}$  are feedback gains to be designed.  $c > 0$  is the coupling strength,  $a_{ij} \geq 0$  are entries of the adjacency matrix  $A = (a_{ij})_{N \times N} \in \mathbb{R}^{N \times N}$  with  $a_{ii} = 0$ , which represents the topology of the network. Here, we are only interested in undirected networks, i.e., if there is a connection between nodes  $i$  and  $j$  ( $i \neq j$ ), then  $a_{ij} = a_{ji} = 1$ ; otherwise  $a_{ij} = a_{ji} = 0$ .  $\hat{v}_j^i \in \mathbb{R}^{n_v}$ ,  $j \in \mathcal{N}_i$  are the states of  $N_i$  estimators  $\mathcal{O}_j^i$  that are built in node  $i$  with the form of

$$\mathcal{O}_j^i: \begin{cases} \dot{\hat{v}}_j^i(t) = H \hat{v}_j^i(t), & t \in [t_{k_j}^j, t_{k_j+1}^j), j \in \mathcal{N}_i \\ \hat{v}_j^i(t_{k_j}^{j+}) = v_j(t_{k_j}^j), & \text{whenever } r_j(t, v_j, \hat{v}_j^j, \hat{z}_j) > 0, \end{cases} \quad (3)$$

where  $\mathcal{N}_i = \mathcal{N}_i \cup \{i\}$ ;  $\mathcal{N}_i = \{j \in \{1, 2, \dots, N\} \mid a_{ij} > 0\}$  is the neighbour index set of node  $i$ ;  $N_i$  is the cardinality of the set  $\mathcal{N}_i$ . The increasing time sequence  $\{t_{k_j}^j\}$ ,  $k_j \in \mathbb{Z}^+$  which determines when node  $j$  communicates to its neighbours, is decided by the event-triggering function  $r_j(\cdot, \cdot, \cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_v} \rightarrow \mathbb{R}$ , i.e.,

$$t_{k_j+1}^j = \inf \left\{ t \geq t_{k_j}^j \mid r_j(t, v_j, \hat{v}_j^j, \hat{z}_j) > 0 \right\} \quad (4)$$

where  $t_0^j = t_0$ ,  $\hat{z}_j = \sum_{l \in \mathcal{N}_j} (\hat{v}_l^j - \hat{v}_j^j)$ . These estimators are reinitialized at each time when node  $i$  receives the sampled state of subsystem  $v_j$  from its neighbours, and hence, provide node  $i$  with an estimate of the current state of each subsystem  $v_j$  from node  $j$  during the time interval  $[t_{k_j}^j, t_{k_j+1}^j)$ ,  $j \in \mathcal{N}_i$ .

We use the following assumptions throughout the paper.

- A1. The dynamical network (1) is connected, i.e., the adjacency matrix  $A$  is irreducible;
- A2. There is no time delay for the sampling and sending executions, i.e., the time  $t_{k_i}^i$  represents both the  $k_i$ th sampling time instant and the  $k_i$ th time when node  $i$  broadcasts its sampled value  $v_i(t_{k_i}^i)$  to its neighbours;
- A3. The communication network is under an ideal circumstance, i.e., there are no time delays or data dropouts in communication, i.e., node  $j$  receives  $v_i(t_{k_i}^i)$  instantaneously at  $t = t_{k_i}^i$ .

Under Assumptions A2 and A3, all the estimators  $\mathcal{O}_i^j$  for each  $j \in \mathcal{N}_i$  will be reinitialized simultaneously using the value  $v_i(t_{k_i}^i)$  at  $t = t_{k_i}^{i+}$ , i.e.,  $\hat{v}_i^j(t_{k_i}^{i+}) = v_i(t_{k_i}^i)$ . This together with (3) and  $t_0^j = t_0$  leads to

$$\hat{v}_i^j(t) = \hat{v}_i^i(t) \quad \forall j \in \mathcal{N}_i, t \geq t_0,$$

which implies that all estimators  $\mathcal{O}_i^j$ ,  $j \in \mathcal{N}_i$  that are built in different nodes to estimate the same subsystem  $v_i$  have the same state response all the time. To simplify the analysis, we will not distinguish these estimators  $\mathcal{O}_i^j$ , and use  $\hat{v}_i$  to replace  $\hat{v}_i^j$  in the sequel. Therefore, network (1) with (2) and

(3) can be simplified as

$$\dot{x}_i = \bar{A}_i x_i + \bar{E}_i v_i \quad (5a)$$

$$y_i = C_i x_i \quad (5b)$$

$$\dot{v}_i = H v_i - c \sum_{j=1}^N l_{ij} \hat{v}_j(t), \quad (5c)$$

$$\mathcal{O}_i: \begin{cases} \hat{v}_i = H \hat{v}_i, & t \in [t_{k_i}^i, t_{k_i+1}^i) \\ \hat{v}_i(t^+) = v_i(t), & \text{whenever } r_i(t, v_i, \hat{v}_i, \hat{z}_i) > 0, \end{cases} \quad (5d)$$

where  $\bar{A}_i = A_i + B_i K_{x_i}$ ,  $\bar{E}_i = E_i + B_i K_{v_i}$ . The matrix  $L = (l_{ij})_{N \times N} \in \mathbb{R}^{N \times N}$  is the Laplacian matrix associated with the adjacency matrix  $A$ , which is defined as follows

$$l_{ij} = \begin{cases} \sum_{z=1}^N a_{iz}, & i = j \\ -a_{ij}, & i \neq j. \end{cases} \quad (6)$$

*Remark 2:* Of course, the problem becomes more complicated if the communication network is not ideal, and deserves more attention.

*Definition 1:* Let  $\xi(t; \xi_0) = (\xi_1(t; \xi_0)^\top, \xi_2(t; \xi_0)^\top, \dots, \xi_N(t; \xi_0)^\top)^\top \in \mathbb{R}^{\bar{N}}$  be a solution to network (5) with the initial condition  $\xi_0 = (\xi_{10}^\top, \xi_{20}^\top, \dots, \xi_{N0}^\top)^\top$  and  $\xi_{i0} = \xi_i(t_0)$ , where  $\xi_i = (x_i^\top, v_i^\top, \hat{v}_i^\top)^\top$  and  $\bar{N} = \sum_{i=1}^N n_i + 2Nn_v$ . Then, we say network (5) achieves *output synchronization asymptotically*, if  $\xi(t; \xi_0)$  exists for every initial condition  $\xi_0 \in \mathbb{R}^{\bar{N}}$  and for all  $t > t_0$  such that

$$\lim_{t \rightarrow \infty} \|y_i(t) - y_j(t)\| = 0, \quad (7)$$

for all  $i, j = 1, 2, \dots, N$ .

We also use the following assumptions to specify each node dynamics which have been extensively used in the literature of output regulation [16].

A4.  $H$  has no eigenvalues with negative real parts;

A5. The pair  $(A_i, B_i)$  for all  $i = 1, 2, \dots, N$  is stabilizable.

*Remark 3:* Similar to [16], Assumption A4 is made only for convenience and loses no generality. Actually, if the output synchronization problem is solvable by controllers (2) under Assumption A4, then it is also solvable by the same controllers if Assumption A4 is violated.

We give a lemma which will be used in the proof of the main results to finish this section.

*Lemma 1 ([15]):* Consider the nonlinear system

$$\dot{x} = f(x, t), \quad (8)$$

where  $f: \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$  is continuous. Suppose that for any  $M > 0$ , there exists  $\eta_M$ , such that

$$\|f(x, t)\| \leq \eta_M, \forall t \geq 0 \text{ and } \|x\| < M. \quad (9)$$

Then, if there exist a nonnegative bounded function  $g(t)$  defined on  $\mathbb{R}^+$ , a smooth function  $V(x, t): \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}$  and  $\mathcal{K}$  functions  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  satisfying

$$\alpha_1(\|x\|) \leq V(x, t) \leq \alpha_2(\|x\|), \quad (10)$$

$$\dot{V}(x, t) \leq -\alpha_3(\|x\|), \text{ whenever } \|x\| \geq g(t) \quad (11)$$

$$\lim_{t \rightarrow \infty} g(t) = 0, \quad (12)$$

then  $x = 0$  of system (8) is asymptotically stable.

### III. DISTRIBUTED EVENT-TRIGGERED CONTROL

To achieve the main goal of the paper, define the error vector  $e_i(t) = \hat{v}_i(t) - v_i(t)$ , and denote  $v = (v_1^\top, v_2^\top, \dots, v_N^\top)^\top$ ,  $e = (e_1^\top, e_2^\top, \dots, e_N^\top)^\top$ . Then, (5c) can be rewritten as follows

$$\dot{v} = (I_N \otimes H - cL \otimes I_{n_v})v - (cL \otimes I_{n_v})e. \quad (13)$$

Since the matrix  $L$  is irreducible, symmetric, and has zero row sums (6), there always exists a unitary matrix  $\Psi = (\psi_1, \psi_2, \dots, \psi_N) \in \mathbb{R}^{N \times N}$  with  $\psi_i = (\psi_{i1}, \psi_{i2}, \dots, \psi_{iN})^\top \in \mathbb{R}^N$  and  $\Psi^\top \Psi = I_N$  such that

$$\Psi^\top L \Psi = \Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N),$$

where  $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_N$ , and  $\lambda_1 = 0$  with algebraic multiplicity one. Furthermore, we can choose  $\psi_1 = \frac{1}{\sqrt{N}}(1, 1, \dots, 1)^\top$  for  $\lambda_1 = 0$ , and this leads to

$$\sum_{j=1}^N \psi_{ij} = 0 \text{ for all } i = 2, 3, \dots, N.$$

Let  $\Phi = (\psi_2, \psi_3, \dots, \psi_N) \in \mathbb{R}^{N \times (N-1)}$ ,  $\Lambda_1 = \Phi^\top L \Phi = \text{diag}\{\lambda_2, \lambda_3, \dots, \lambda_N\}$  and  $\bar{\Phi} = \Phi \otimes I_{n_v}$ . Defining  $\bar{v} = \bar{\Phi}^\top v$ , we have

$$\begin{aligned} \dot{\bar{v}} &= \bar{\Phi}^\top (I_N \otimes H - cL \otimes I_{n_v})v - \bar{\Phi}^\top (cL \otimes I_{n_v})e \\ &= (I_{N-1} \otimes H - c\Lambda_1 \otimes I_{n_v})\bar{v} - (c\Lambda_1 \otimes I_{n_v})\bar{\Phi}^\top e, \end{aligned} \quad (14)$$

where we use properties  $\bar{\Phi}^\top (I_N \otimes H) = (I_{N-1} \otimes H)\bar{\Phi}^\top$  and  $(cL \otimes I_{n_v})(I_{Nn_v} - \bar{\Phi}\bar{\Phi}^\top) = 0$ , which are supported by the fact  $\bar{\Phi}^\top \Phi = I_{N-1}$ ,  $L1_N = 0$ ,  $I_N - \Phi\Phi^\top = \frac{1}{N}1_{N \times N}$ . Let  $\bar{H} = I_{N-1} \otimes H - c\Lambda_1 \otimes I_{n_v} = \text{diag}\{H_2, H_3, \dots, H_N\}$  with  $H_i = H - c\lambda_i I_{n_v}$ , and  $\bar{\Lambda} = c\Lambda_1 \otimes I_{n_v}$ . Then (14) can be written as

$$\dot{\bar{v}} = \bar{H}\bar{v} - \bar{\Lambda}\bar{\Phi}^\top e. \quad (15)$$

*Lemma 2:* Under Assumptions A1 to A5, let the feedback gain  $K_{x_i}$  be such that  $A_i + B_i K_{x_i}$  is Hurwitz, i.e. all its eigenvalues have negative real parts. If there exist matrices  $X_i$ ,  $U_i$  satisfying

$$X_i H = A_i X_i + B_i U_i + E_i, \quad (16)$$

$$C_i X_i = F, \quad (17)$$

and if

$$\lim_{t \rightarrow \infty} \|\bar{v}(t)\| = 0, \quad (18)$$

then network (5) achieves output synchronization asymptotically with the feedback gain  $K_{v_i}$  given by

$$K_{v_i} = U_i - K_{x_i} X_i. \quad (19)$$

*Proof:* Let  $\bar{x}_i = x_i - X_i v_i$ , we have

$$\dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + cX_i \sum_{j=1}^N l_{ij} v_j + cX_i \sum_{j=1}^N l_{ij} e_j, \quad (20)$$

$$y_i = C_i \bar{x}_i + F v_i.$$

It is shown in [8] that condition (18) leads to

$$\lim_{t \rightarrow \infty} \|v_i(t) - v_j(t)\| = 0, \quad (21)$$

which further yields

$$\lim_{t \rightarrow \infty} \|e_i(t) - e_j(t)\| = 0. \quad (22)$$

Based on (21) and (22), we have

$$\lim_{t \rightarrow \infty} \|cX_i \sum_{j=1}^N l_{ij} v_j\| = 0, \quad \lim_{t \rightarrow \infty} \|cX_i \sum_{j=1}^N l_{ij} e_j\| = 0. \quad (23)$$

Since  $\bar{A}_i = A_i + B_i K_{x_i}$  is Hurwitz, applying Lemma 1 gives system (20) is asymptotically stable, i.e.,

$$\lim_{t \rightarrow \infty} \|\bar{x}_i(t)\| = 0, \quad (24)$$

for all  $i = 1, 2, \dots, N$ . Therefore, we have

$$\lim_{t \rightarrow \infty} \|y_i(t)\| = \lim_{t \rightarrow \infty} \|C_i \bar{x}_i(t) + F v_i(t)\| = \lim_{t \rightarrow \infty} \|F v_i(t)\|, \quad (25)$$

and asymptotic output synchronization of network (5) follows from (21) and (25) directly. ■

Now, we discuss the output synchronization problem of (5) by designing a proper event-triggering rule. In the case where all states of network (5) can be accessed by each node, we can design a centralized event-triggering rule.

*Theorem 1:* Under Assumptions A1 to A5, let the feedback gain  $K_{x_i}$  and  $K_{v_i}$  be given in Lemma 2. If there exist positive definite matrices  $P_i$  such that

$$(H - c\lambda_i I_{n_v})^\top P_i + P_i (H - c\lambda_i I_{n_v}) = -2I_{n_v}, \quad (26)$$

and matrices  $X_i$  and  $U_i$  satisfying (16) and (17), then network (5) achieves output synchronization asymptotically under the sampling time sequence determined by the centralized event-triggering function  $r(e, v) = \|\bar{\Phi}^\top e\| - \frac{\delta}{\alpha} \|\bar{\Phi}^\top v\|$ , i.e.,

$$t_{k+1} = \inf\{t > t_k \mid r(e, v) > 0\}, \quad (27)$$

where  $\delta \in (0, 1)$ ,  $\alpha = \max_{i=2,3,\dots,N} \{-c\lambda_i \|P_i\|\}$ , and  $\lambda_i$ ,  $i = 2, 3, \dots, N$  are non-zero eigenvalues of the Laplacian matrix  $L$ . Moreover, no Zeno behaviour occurs in (5) for all  $t \geq t_0$ .

*Proof:* First, we claim that with (27), there exists a  $\tau^* > 0$  such that  $\tau_k = t_{k+1} - t_k \geq \tau^*$ ,  $\forall k \in \mathbb{Z}$ . Similar to the proof of Theorem 2 in [10], we get

$$\frac{d}{dt} \frac{\|\bar{\Phi}^\top e\|}{\|\bar{\Phi}^\top v\|} \leq a \frac{\|\bar{\Phi}^\top e\|^2}{\|\bar{\Phi}^\top v\|^2} + 2(a+b) \frac{\|\bar{\Phi}^\top e\|}{\|\bar{\Phi}^\top v\|} + a, \quad (28)$$

where  $a = \|\bar{\Lambda}\|$  and  $b = \|I_{N-1} \otimes H\|$ . Consider the differential equation:

$$\dot{\phi} = a\phi^2 + 2(a+b)\phi + a, \quad (29)$$

then we conclude that the inter-execution intervals  $\tau_k$  is lower bounded by the time during which  $\phi$  evolves from 0 to  $\frac{\delta}{\alpha}$ , i.e.,  $\phi(\tau^*, 0) = \frac{\delta}{\alpha}$ . Such a  $\tau^*$  can be obtained by solving the differential equation (29), namely

$$\tau^* = \frac{1}{2ac} \ln \left| \frac{\delta + \alpha c_1 - \alpha c}{\delta + \alpha c_1 + \alpha c} \right| + \frac{c_0}{a} > 0 \quad (30)$$

with  $c_1 = \frac{a+b}{a}$ ,  $c = \sqrt{\frac{(a+b)^2 - a^2}{a}} > 0$  and  $c_0 = -\frac{1}{2c} \ln \left| \frac{c_1 - c}{c_1 + c} \right|$ . Thus, no Zeno behaviour occurs in network (5) under event-triggering rule (27) for all  $t \geq t_0$ .

Select the following Lyapunov function candidate

$$V = \bar{v}^\top P \bar{v}, \quad (31)$$

where  $P = \text{diag}\{P_2, P_3, \dots, P_N\}$  and  $P_i$ ,  $i = 2, 3, \dots, N$  are positive definite matrix solutions of (26). Then along the trajectories of system (15), one has

$$\dot{V} = \dot{\bar{v}}^\top P \bar{v} + \bar{v}^\top P \dot{\bar{v}} \leq -2\|\bar{v}\|^2 - 2\|cP\bar{\Lambda}\| \|\bar{v}\| \|\bar{\Phi}^\top e\|. \quad (32)$$

The event-triggering rule (27) ensures that  $\|\bar{\Phi}^\top e\| \leq \frac{\delta}{\alpha} \|\bar{\Phi}^\top v\| = \frac{\delta}{\alpha} \|\bar{v}\|$  for all  $t \geq t_0$ . This together with (32) makes the following inequality hold

$$\dot{V} \leq -2(1 - \delta) \|\bar{v}\|^2. \quad (33)$$

Therefore, the equilibrium point  $\bar{v} = 0$  of system (15) is asymptotically stable, i.e.,  $\lim_{t \rightarrow \infty} \|\bar{v}(t)\| = 0$ . Applying Lemma 2 proves the theorem. ■

*Remark 4:* For any given  $H$ , there always exists a  $\bar{d} > 0$  such that  $H - dI_{n_v}$  is Hurwitz for all  $d \geq \bar{d}$ . Therefore, condition (26) can be simply checked by

$$c \geq \frac{\bar{d}}{\lambda_2}, \quad (34)$$

which is in accordance with the results in [17] for networks with identical nodes connected continuously.

In practice, a centralized event-triggering rule is usually hard to implement because it may be costly and time consuming to gather global information for the design purpose. Therefore, a distributed event-triggering rule for each node which only relies on information that it can get is desirable.

*Theorem 2:* Under Assumptions A1 to A5, let  $K_{x_i}$  and  $K_{v_i}$  be given in Lemma 2. If there exist matrices  $X_i$ ,  $U_i$  and positive definite matrices  $P_i$  such that (16), (17) and (26) are satisfied, then network (5) achieves output synchronization asymptotically under the distributed event-triggering rule

$$t_{k+1}^i = \inf\{t \geq t_k^i \mid r_i(t, e_i, \hat{z}_i) > 0\}, \quad (35)$$

where  $r_i(t, e_i, \hat{z}_i) = \|e_i\| - \rho \sqrt{\|\hat{z}_i\|^2 + e^{-2\gamma t}}$ ;  $\rho = \frac{\delta}{\lambda_N(\alpha + \delta)} > 0$ ;  $\lambda_N > 0$  is the largest eigenvalue of  $L$ ;  $\gamma$  is a positive constant such that  $\gamma < \lambda_{\min}$  with  $\lambda_{\min} = -\max_{i \in \{2, 3, \dots, N\}} \{\text{Re}(\lambda(H_i))\}$ ;  $\alpha$  and  $\delta$  are given in Theorem 1. Moreover, no Zeno behaviour occurs in (5) for all  $t \geq t_0$ .

*Proof:* We divide the proof into two steps: 1) to show the existence of a lower bound on the inter-execution intervals for event-triggering rule (35); and 2) to prove asymptotic output synchronization of network (5).

To prove 1), we consider the event-triggering rule

$$t_{k+1}^i = \inf\{t \geq t_k^i \mid \|e_i\| > \rho e^{-\gamma t}\}. \quad (36)$$

By using the method proposed in [18], we can get an upper bound on  $\|\bar{v}\|$ , that is

$$\|\bar{v}\| \leq k_0 e^{-\lambda_{\min}(t-t_0)} \|\bar{v}(t_0)\| + k_0 \int_{t_0}^t e^{-\lambda_{\min}(t-\theta)} \|\bar{\Lambda}\| \|\bar{\Phi}^\top e(\theta)\| d\theta, \quad (37)$$

where  $k_0 = \|U_T\| \|U_T^{-1}\|$ , and  $U_T$  is a nonsingular matrix such that  $U_T \bar{H} U_T^{-1} = D$  with  $D$  being the diagonal matrix

composed of the eigenvalues of  $\bar{H}$ . The event-triggering rule (36) guarantees that  $\|e_i\| < \rho e^{-\gamma}$  for all  $t \geq t_0$ , which together with the property  $\|\bar{\Phi}\| = 1$  gives

$$\|\bar{\Phi}^\top e\| \leq \|\bar{\Phi}\| \|e\| = \|e\| < \sqrt{N} \rho e^{-\gamma}.$$

Substituting the above inequality into (37) gives

$$\|\bar{v}\| \leq k_0 \|\bar{v}(t_0)\| e^{\lambda_{\min} t_0} e^{-\lambda_{\min} t} + \frac{k_0 \rho \sqrt{N} \|\bar{\Lambda}\|}{\lambda_{\min} - \alpha} e^{-\gamma}. \quad (38)$$

In addition, the dynamics of  $e_i$  can be rewritten as

$$\dot{e}_i = H e_i + \Gamma \hat{z}_i, \quad (39)$$

which gives

$$\begin{aligned} \|\dot{e}_i\| &\leq \|H\| \|e_i\| + \|\Gamma\| \|\hat{z}_i\| \\ &\leq \rho (\|H\| + \lambda_N \sqrt{N} \|\Gamma\|) e^{-\gamma} + \|\Gamma\| \|z\|, \end{aligned} \quad (40)$$

where  $z = (A \otimes I_{n_v})v$ , and  $\|A \otimes I_{n_v}\| = \lambda_N$ . Moreover,

$$\begin{aligned} \|z\|^2 &= v^\top (A^\top \otimes I_{n_v}) v \leq \lambda_N^2 v^\top (U^2 \otimes I_{n_v}) v \\ &= \lambda_N^2 \|\bar{\Phi}^\top v\|^2 = \lambda_N^2 \|\bar{v}\|^2, \end{aligned} \quad (41)$$

which comes from  $A^2 \leq \lambda_N^2 U^2$  and  $U^2 = U = \bar{\Phi} \bar{\Phi}^\top$ . Therefore, the inequality (40) becomes

$$\|\dot{e}_i\| \leq \rho (\|H\| + \lambda_N \sqrt{N} \|\Gamma\|) e^{-\gamma} + \lambda_N \|\Gamma\| \|\bar{v}\|. \quad (42)$$

From (42) and (38), for all  $t \in [t_{k_i}^i, t_{k_i+1}^i)$  and  $k \in \mathbb{Z}$ , we have

$$\|\dot{e}_i\| \leq k_1 e^{-\lambda_{\min} t} + k_2 e^{-\gamma} \leq k_1 e^{-\lambda_{\min} t_{k_i}^i} + k_2 e^{-\gamma_{k_i}^i},$$

where  $k_1 = k_0 \|\Gamma\| \|\bar{v}(t_0)\| e^{-\lambda_{\min} t_0}$  and  $k_2 = \frac{k_0 \rho \sqrt{N} \|\bar{\Lambda}\|}{\lambda_{\min} - \gamma} + \rho (\|H\| + \lambda_N \sqrt{N} \|\Gamma\|)$ . Thus, we get

$$\begin{aligned} \|e_i(t)\| &= \left\| \int_{t_{k_i}^i}^t \dot{e}_i(\theta) d\theta \right\| \leq \int_{t_{k_i}^i}^t \|\dot{e}_i(\theta)\| d\theta \\ &\leq \left( k_1 e^{-\lambda_{\min} t_{k_i}^i} + k_2 e^{-\gamma_{k_i}^i} \right) (t - t_{k_i}^i) \\ &\leq e^{-\gamma_{k_i}^i} (k_1 + k_2) \tau_{k_i}^i, \end{aligned} \quad (43)$$

where  $\tau_{k_i}^i = t - t_{k_i}^i$ . Similar to [7], we conclude that the inter-execution intervals  $\tau_{k_i}^i \leq t_{k_i+1}^i - t_{k_i}^i$  is lower bounded by the solution  $\tau_i^*$  of the equation  $(k_1 + k_2) \tau_i^* = \rho e^{-\gamma \tau_i^*}$  which is strictly positive. Apparently, the inter-execution intervals of event-triggering rule (35) are greater than those of rule (36) because  $\sqrt{\|\hat{z}_i\|^2 + e^{-2\gamma}} \geq e^{-\gamma}$  with  $\|\hat{z}_i\| \geq 0$ , and thus there also exists a positive lower bound on the inter-execution intervals for (35).

To prove 2), event-triggering rule (35) guarantees that

$$\|e_i\| < \rho \sqrt{\|\hat{z}_i\|^2 + e^{-2\gamma}}, \quad \forall t \geq t_0. \quad (44)$$

Since

$$\sum_{i=1}^N (\|\hat{z}_i\|^2 + e^{-2\gamma}) = \|\hat{z}\|^2 + N e^{-2\gamma} \leq \left( \|\hat{z}\| + \sqrt{N} e^{-\gamma} \right)^2,$$

we have

$$\begin{aligned} \|e\| &< \rho \left( \|\hat{z}\| + \sqrt{N} e^{-\gamma} \right) \\ &\leq \rho (\|z\| + \lambda_N \|e\| + \sqrt{N} e^{-\gamma}). \end{aligned} \quad (45)$$

As  $\rho = \frac{\delta}{\lambda_N(\alpha + \delta)}$ , the inequality (45) is equivalent to

$$\|e\| < \frac{\delta}{\alpha \lambda_N} \left( \|z\| + \sqrt{N} e^{-\gamma} \right). \quad (46)$$

Using (41) with  $\|\bar{\Phi}^\top e\| \leq \|e\|$  leads to

$$\|\bar{\Phi}^\top e\| < \|e\| < \frac{\delta}{\alpha} \left( \|\bar{\Phi}^\top v\| + \frac{\sqrt{N} e^{-\gamma}}{\lambda_N} \right). \quad (47)$$

Select the Lyapunov function candidate (31). Then, its derivative along system (15) satisfies

$$\begin{aligned} \dot{V} &\leq -2\|\bar{v}\|^2 + 2\alpha \|\bar{v}\| \|\bar{\Phi}^\top e\| \\ &\leq -2(1 - \delta) \|\bar{v}\|^2 + 2\rho_1 \|\bar{v}\| e^{-\gamma} \\ &= -2(1 - \delta - \delta_1) \|\bar{v}\|^2 - 2\|\bar{v}\| (\delta_1 \|\bar{v}\| - \rho_1 e^{-\gamma}) \\ &\leq -2(1 - \delta - \delta_1) \|\bar{v}\|^2, \text{ when } \|\bar{v}\| \geq \rho_2 e^{-\gamma}, \end{aligned} \quad (48)$$

where  $0 < \delta_1 < 1 - \delta$ ,  $\rho_1 = \frac{\delta \sqrt{N}}{\lambda_N} > 0$ ,  $\rho_2 = \frac{\delta \sqrt{N}}{\delta_1 \lambda_N} > 0$ . Let  $g(t) = \rho_2 e^{-\gamma}$ . We have  $\lim_{t \rightarrow \infty} g(t) = 0$ . Applying Lemma 1 gives that the equilibrium  $\bar{v} = 0$  of system (15) is asymptotically stable. Applying Lemma 2 proves the theorem. ■

*Remark 5:* This paper extends results proposed in [10] to a more general case where the network has non-identical nodes. Apparently, if  $A_i = H$ ,  $B_i = \Gamma$ ,  $E_i = 0$  and  $u_{x_i} = c \sum_{j \in \mathcal{N}_i} a_{ij} (\hat{x}_j - \hat{x}_i)$ , then network model (1) reduces to that investigated in [10], therefore, results proposed in the paper contain those in [10] as special cases.

#### IV. AN EXAMPLE

This section gives an example to show the effectiveness of the proposed control. Suppose that the network has 10 nodes, and the parameters are given as follows:  $c = 1$ ,

$$A_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -d_i & -a_i \end{pmatrix}, \quad E_i = E = \begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B_i = \begin{pmatrix} 0 & 0 & b_i \end{pmatrix}^\top, \quad C_i = C = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & -0.5 \\ 0.5 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix},$$

where  $a_i = -0.8 - 0.2 \times i$ ,  $b_i = 0.5 + 0.5 \times i$ ,  $d_i = 3 - 0.2 \times i$ ,  $i = 1, 2, \dots, 10$ . By calculation, we get  $X_i$ ,  $U_i$  and  $F$  as follows such that conditions (16) and (17) are satisfied

$$X_i = X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$U_i = \begin{pmatrix} 0 & \frac{d_i}{b_i} \end{pmatrix}^\top, \quad F = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

Moreover,  $(A_i, B_i)$  are stabilizable, and we choose  $K_{x_i}$  such that  $\bar{A}_i$  has eigenvalues  $-1, -2$  and  $-3$  in the simulation.

For centralized event-triggering rule (27), Figure 1 gives the output of the network, the state of the exosystem and the event time with  $\delta = 0.9$  and  $\alpha = 1$ , which shows that the network achieves output synchronization asymptotically. By (30), we can calculate a lower bound on the inter-execution interval  $\tau^* = 0.075$ , but the minimum inter-execution interval during the simulation time is  $\tau^* = 0.1705$ .

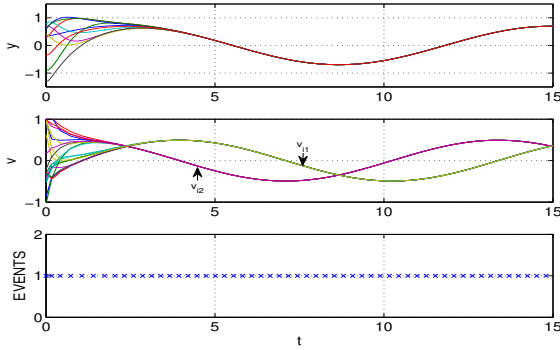


Fig. 1. Simulation for centralized event-triggering rule.

For distributed event-triggering rule (35), Figure 2 shows the output of the network, the state of exosystem  $v$  and event times of each node, where  $\delta = 0.9$ ,  $\alpha = 1$ ,  $\lambda_N = 6.1518$ ,  $\lambda_{\min} = 0.2972$ , and  $\gamma = 0.29$ . The minimum inter-execution interval  $\tau_i^*$  for each node for  $t$  from 0s to 15s is given in Table 1.

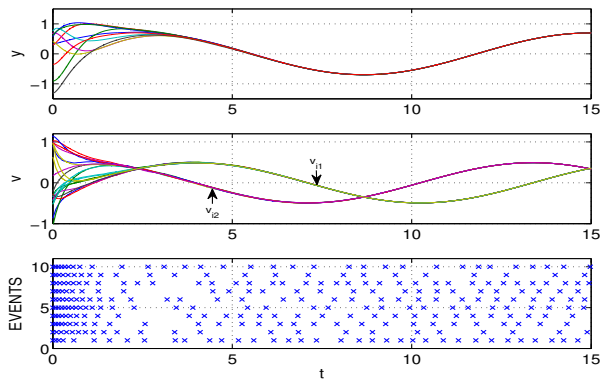


Fig. 2. Simulation for the distributed event-triggering rule.

TABLE I  
THE MINIMUM INTER-EXECUTION INTERVAL

	Node 1	Node 2	Node 3	Node 4	Node 5
$\tau_i^*$	0.0792	0.1031	0.0795	0.0794	0.0793
	Node 6	Node 7	Node 8	Node 9	Node 10
$\tau_i^*$	0.0792	0.0793	0.0922	0.0792	0.0790

## V. CONCLUSION

This paper has investigated the output synchronization problem of a class of dynamical networks by designing a distributed event-triggered controller. By applying output regulation theory, we have extended results proposed in [10] to a more general case where the network has non-identical nodes. A distributed event-triggering rule for each node has been explored, which only relies on the limited information from its neighbours and states of estimators introduced in it. It has been shown that the network achieves output synchronization asymptotically with the proposed event-triggering rule, and no Zeno behaviour occurs.

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